Growth behavior of helical cellular automata

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(Received 19 March 2003; published 29 October 2003)

A helical cellular automata (HCA) model constructed on a two-dimensional grid of cells with a helical structure is presented and the pattern formation of this model studied by numerous computer simulations. It is found that the evolutions of the HCA are sensitive to the circumference of the helix p. With various p, the initial growth of the model generates various patterns ranging from Sierpinski triangle gasket, complex textured pattern, to lateral quasiperiodic structure. A sudden transition from regular fractal to compact pattern occurs near the point where p is equal to a positive integer power of 2. With increasing height of the patterns (increasing growth time), the model also exhibits different growth behaviors in the vertical direction for various p, including the formation of regular periodic patterns and the evolution from initial regular patterns to eventual random structures. Fractal dimension analysis is used to characterize these different evolution processes quantitatively.

DOI: 10.1103/PhysRevE.68.047104 PACS number(s): 05.10.−a, 82.20.Wt

It is generally believed that many complex structures and processes in nature are originated from the simple interaction of large numbers of basic connected components. Cellular automata (CA) defined by simple local rules can be viewed as prototypical models to simulate the complex behavior of the local dynamical systems [1–5]. Previously, we constructed a circular cellular automata (CCA) model [6] grown on a square lattice with the ability to generate fractals and show chaotic behavior. Contrary to the parallel updated rule in classical CA, in the CCA model different sequences for selecting sites are applied, and the growth of a selected site is determined by using an intermediate crowded condition [7,8]. In this Brief Report, we introduce a helical cellular automata (HCA) model grown on a discrete helical structure. This model is motivated from the structures of the helical cylinderlike crystals observed in some biological systems [9,10] and the geometries of the helical carbon and noncarbon nanotubes [11,12], which have been studied intensively in recent years. Using a simple growth rule, the model generates various patterns, from regular fractal to quasiperiodic structures.

A HCA is a two-dimensional, binary valued, nearest-neighbor interaction growth model. The model is constructed on a cylinder, and the grid of cells is designed to have a helical structure with a fixed unit pitch. Figure 1 is a schematic illustration of this model. It can be seen that the circumference p (the length of a complete turn) of the helix is 18 units. In Fig. 1 we denote the sites grown and not by the dark dots and the open dots, respectively. The growth rule of the model is the following: Starting from an initial seed, the sites are selected successively along the circular helix. A selected site can grow (becomes a dark dot) only when just one nearest site is a dark dot among its four nearest sites (intermediate crowded condition). In Fig. 1(a) the dark dots in the first period of the helix (from site 1 to 18) are taken as the seed. The site 19 cannot be grown because it has two nearest dark dots (sites 1 and 18); the next selected site 20 has only one nearest dark dot (site 2), therefore it becomes a new grown site; as the procedure continues, the pattern grows upward successively. To show the pattern and analyze its dimension, the obtained pattern on the cylinder is mapped approximately to the corresponding rectilinear grid [see Fig. 1(b)] with the size $p \times h$, where p and h are the circumference and the height of the original pattern, respectively.

We have studied the spatial pattern formation of the model for different circumference p, and found that as p increases, the evolution of the pattern with a height h shows

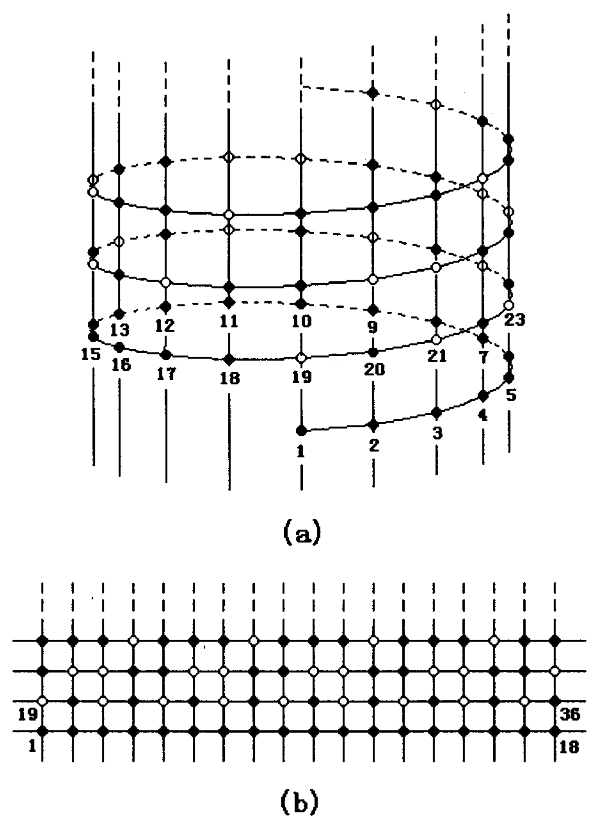

FIG. 1. Schematic illustration of the HCA model.

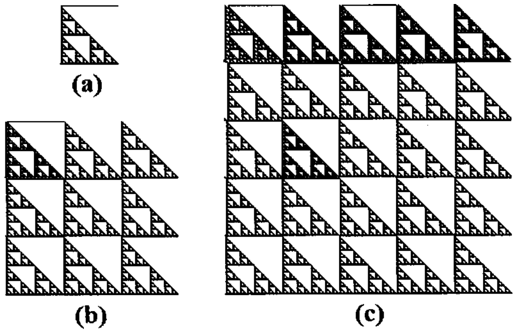

FIG. 2. The first three patterns of $n=6$ pattern sequences with odd p, (a) $p=2^6+1=65$, (b) $p=3\times 2^6+1=193$, (c) $p=5\times 2^6+1=321$.

an interesting recursive behavior. The results of numerous computer simulations can be described as follows.

(1) *Pattern sequences*. When $p=m\times 2^n$ or $m\times 2^n+1$, where n is a positive integer, and m an odd number, the pattern has a regular structure. For a certain n, with $m=1$, 3, 5, 7, etc., the model generates two similar pattern sequences in the increasing p order: 2^n, 3×2^n, $5\times 2^n,\ldots$, (called as even p pattern sequence) and 2^n+1, $3\times 2^n+1$, $5\times 2^n+1,\ldots$, [odd p pattern sequence]. In the even p [odd p] pattern sequence, the first pattern with the size $2^n\times 2^n$ $[(2^n+1)\times(2^n+1)]$ has a structure similar to the Sierpinski triangle gasket, and any other pattern with the size $(m\times 2^n)\times(m\times 2^n)$ $[(m\times 2^n+1)\times(m\times 2^n+1)]$ $(m\neq 1)$ is composed of $m\times m$ elementary pattern blocks which have the similar structure as the first one of the pattern sequence. Figures 2(a)–2(c) show the first three patterns of the odd p sequences for $n=6$. In Fig. 2(a), if cutting out the sites in the first row and the first column (the first column contains only one occupied site), the remaining pattern with the size $2^6\times 2^6$ is a regular Sierpinski gasket. Figures 2(b) and 2(c) are 3×3 and 5×5 arrays consisting of the small pattern blocks, respectively. As can be seen, these pattern block arrays show a lateral quasiperiodicity.

(2) *Pattern transition from fractal to compact*. From $p=2^n+1$, the evolution of the pattern undergoes abrupt change. Figure 3 provides examples of the pattern evolution with increasing p in a narrow range of $p\geqslant 2^8+1$. Figure 3(a) is the first pattern $(p=2^8+1)$ of the odd p pattern sequence for $n=8$, which is structurally equivalent to Fig. 2(a). With an increase in p of only one unit, the simulation generates the pattern shown in Fig. 3(b). The pattern is still regular. As compared with Fig. 3(a), many blank triangle regions in Fig. 3(a) now become completely filled with the grown sites, making the pattern very compact. The same compact pattern can be obtained with $p=2^8-1$. Further increasing p in $p>2^8+1$ region (or decreasing p in $p<2^8-1$ region) will generate the overall homogeneous, but highly textured patterns, meanwhile irregular growth behavior appears as shown in Figs. 3(c) and 3(d), where there are many locally random or amorphous domains immersed inside the regular small pattern block arrays.

As shown above, for different p, the densities of the grown sites of the patterns are obviously different. We can describe this kind of density variation quantitatively by calculating the occupation percentage of pattern η versus p, where η is defined as the ratio of the number of the occupied sites to that of all the sites in a pattern with the height $h=p$. Figure 4 shows the curve of η-p. It can be seen that there are sudden drops of η at the points $p=m\times 2^n$ and $m\times 2^n+1$, and with the increasing exponent n, η decreases. Therefore, the occupation percentages η belonging to different pattern sequences can be distinguished fairly well from the η-p plot. On the other hand, for a certain pattern sequence such as $n=5$ or 6, with increasing m, the values of η have also small variation, and in the case of $n\leqslant 4$ the η even shows an obvious fluctuation. This is due to the appearance of the irregular growth behaviors in some local regions in-

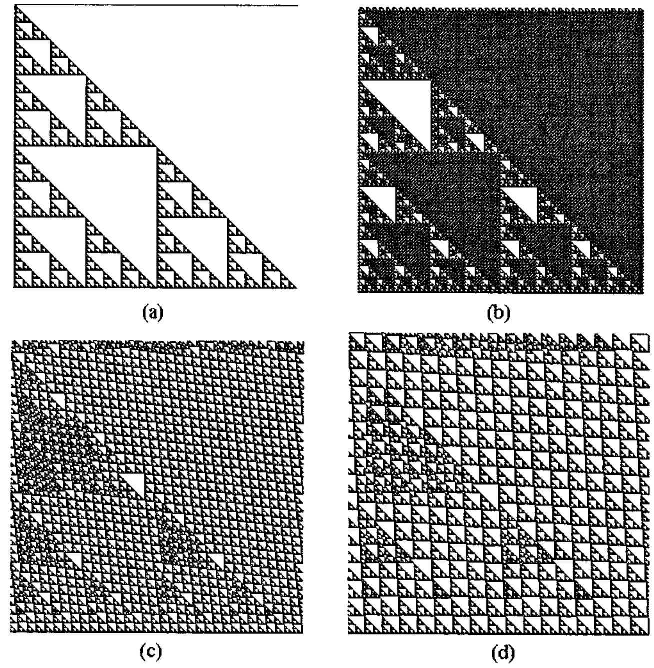

FIG. 3. Evolution of HCA with (a) $p=257$, (b) $p=258$, (c) $p=264$, (d) $p=272$.

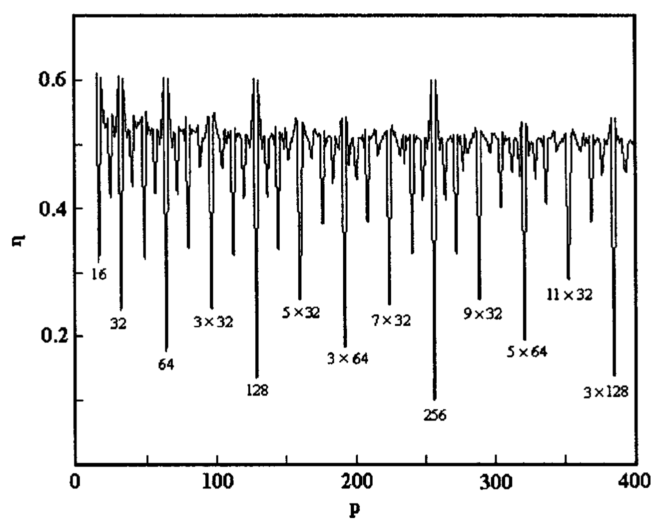

FIG. 4. Dependence of the occupation percentage η on the circumference of the helix p.

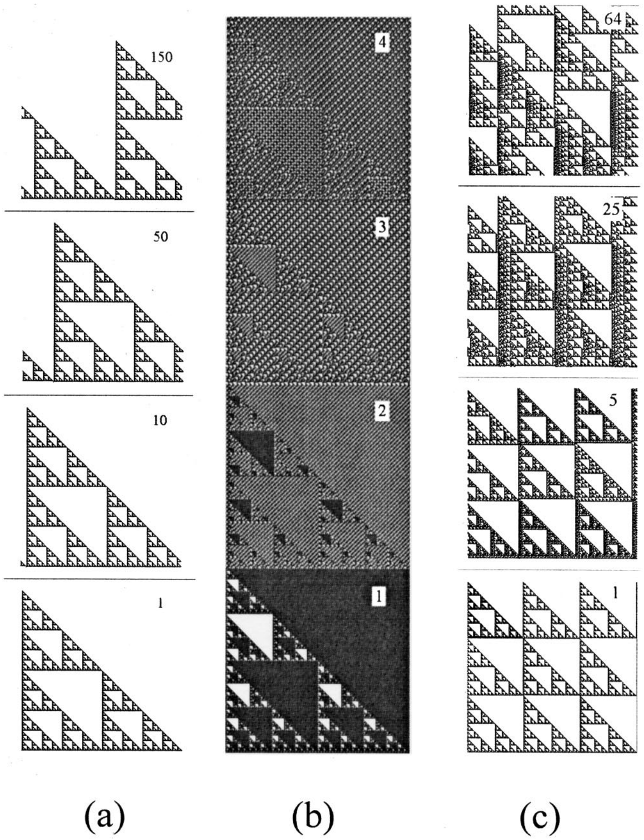

FIG. 5. Pattern evolution with increasing height with (a) $p=257$, (b) $p=258$, and (c) $p=384$. Each block is numbered according to its generated order.

side the pattern block arrays as shown in Figs. 2(b), 2(c), Figs. 3(c) and 3(d), etc. In addition, a strong fluctuation in the occupation percentage occurs at the position near $p=2^n$ or 2^n+1, an indication of the sudden transition from regular fractal growth to the compact growth. The η-p plot clearly exhibits the sensitivity of the model to the variation in the circumference of the helix.

According to the rule of the HCA model, the pattern growth in the vertical direction can be infinite, therefore another important aspect of the model is the long term behavior of a growing pattern. In the following text, the pattern evolution with increasing height (or increasing growth time) is shown for several typical circumferences of the helix p.

Further growth starting from the regular Sierpinski triangle gasket with $p=2^8+1$ is shown in Fig. 5(a). Considering the lateral quasiperiodicity of the system, with increasing height the growth process successively generates triangle patterns similar to the initial one. To describe the evolution process of the pattern conveniently, every triangle pattern is numbered according to its order of appearance. Figure 5(b) shows the upward evolution of the compact pattern ($p=2^8+2$) with the height h from 1 to $2^8\times 4$. The pattern consists of four small square blocks denoted by number 1 to 4. As seen clearly, these pattern blocks with the same size $(2^8+2)\times 2^8$ exhibit obviously different texture. Further increasing h, the simulation will generate complex, but still highly textured pattern. The evolution of the square blocks of size

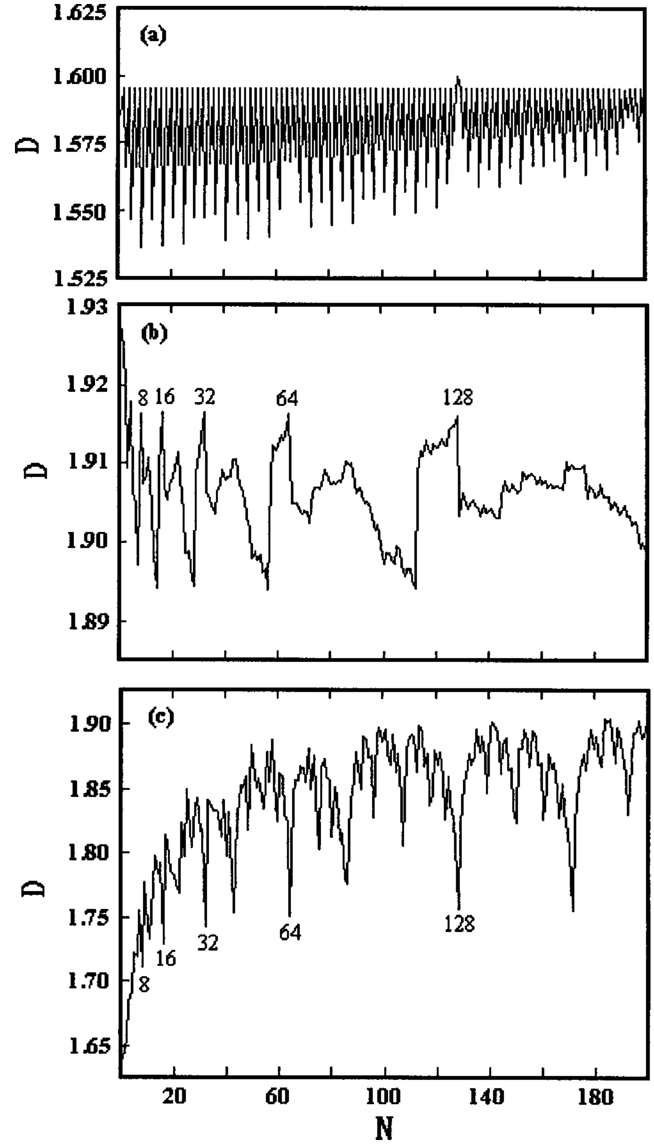

FIG. 6. Dependence of fractal dimension of the pattern blocks on the ordinal number N. (a), (b), and (c) correspond to the pattern evolution shown in Figs. 5(a)–5(c), respectively.

384×384 with increasing height is shown in Fig. 5(c). We also divide the pattern into the numbered pattern blocks with the size 384×384 during growth. At early stage of the growth, the pattern has still a regular structure, but as the height increases, more and more irregular growth behaviors appear. Different to the previous two examples, the initial regular triangle array evolves to a completely random structure eventually composed of many big or small empty triangles. From the examples shown in Figs. 5(b) and 5(c), no overall vertical periodicity seems to emerge, at least for the limited height available.

In order to characterize quantitatively the pattern evolution with increasing height, we perform fractal dimension analysis on the numbered pattern blocks in the plane. The dimension of a pattern block is calculated by using the box-counting method. Figures 6(a)–6(c) present the dimension evolution with increasing ordinal number of the blocks N

(increasing height), corresponding to Figs. 5(a)–5(c). As seen from Fig. 6(a), for the regular growth like Fig. 5(a) the *D-N* curve exhibits regular oscillation, and many different blocks have almost the same dimension value. The dimension calculated from the first pattern block in Fig. 5(a) is 1.584 96, which is just equal to the fractal dimension of regular Sierpinski triangle gasket as expected. The *D-N* curve [Fig. 6(b)] obtained from the evolution of the initial compact pattern shows many irregular fluctuations. It is found that the peaks of dimension appear at the positions $N=2^n$. The simulation result indicates that although these dimension peaks have almost the same heights, the corresponding blocks are obviously different. In Fig. 6(c), the *D-N* curve also shows random fluctuations, but on the average, there is a trend of increase of D. Contrary to Fig. 6(b), now sharp drops in D occur at the positions $N=2^n$. These results clearly show that the pattern evolution with increasing height strongly depends on the circumference of the helix p, and different p can lead to completely different growth behaviors.

The present HCA is quite simple. To improve the simulation results as compared with the actual growth experiments [9–12], the model may be developed in the following way. The intermediate crowded condition may be extended to include the second-nearest neighbors. All sites on the circular helix can be rearranged so that each site has six nearest neighbors, then the obtained pattern on the cylinder can be mapped to a hexagonal grid in the plane. The model is deterministic by construction. However, randomness can be rather conveniently incorporated, either by choosing a random initial seed or by assigning a probability to each growth site. Despite its simplicity it seems probable that this model may provide some insights into the pattern formation processes for the helical growth system and may also prove useful in future studies seeking the controlled design of nanostructure morphologies.

To summarize, we have presented a helical CA model and investigated the behavior of the evolutionary patterns of the simple deterministic CA with only one control parameter. The initial growth of the model generates various patterns including Sierpinski triangle gasket, regular compact pattern, and array structure composed of elementary pattern blocks. With increasing height these initial patterns evolve to regular periodic structure, complex textured pattern, or completely random pattern, eventually. The results of the occupation percentage and fractal dimension analyses show that the evolutions of the model are sensitive to the circumference of the helix.

[1] S. Wolfram, *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1987).
[2] R. Fisch, Physica D **45**, 19 (1990).
[3] E. Domany and W. Kinzel, Phys. Rev. Lett. **53**, 447 (1984).
[4] W. Kinzel, Z. Phys. B: Condens. Matter **58**, 229 (1985).
[5] O. Miramontes, R. V. Sole, and B. C. Goodwin, Physica D **63**, 145 (1993).
[6] X. Sun, D. M. Wang, and Z. Q. Wu, Phys. Rev. E **64**, 036105 (2001).
[7] Z. Q. Wu and B. Q. Li, Phys. Rev. E **51**, R16 (1995).
[8] B. Q. Li *et al.*, Europhys. Lett. **30**, 239 (1995).
[9] N. A. Kiselev and F. Ya. Lerner, J. Mol. Biol. **86**, 578 (1974).
[10] D. L. D. Caspar and A. Klug, Cold Spring Harbor Symp. Quant. Biol. **27**, 1 (1962).
[11] S. Iijima and T. Ichihashi, Nature (London) **363**, 603 (1993).
[12] Y. Kondo and K. Takayanagi, Science **289**, 606 (2000).